# Spherical Harmonics and Spectral Model Solution

**Notation**

| | |
|---|---|
| $(x, y, z)$ | eastward, northward, upward coordinates |
| $(u, v, w)$ | velocity vector components corresponding to $(x, y, z)$ |
| $(\varphi, \lambda)$ | latitude, longitude |
| $a$ | radius of the earth |
| $\mu = \sin\varphi$ | sine of latitude used in spherical harmonics |
| $P_n$ | Legendre Polynomial |
| $P_n^m$ | Associated Legendre Function |
| $Y_n^m = P_n^m e^{im\lambda}$ | Spherical Harmonics |
| $\xi_n^m$ | renormalization factor for spherical harmonics |
| $M$ | zonal wavenumber of the harmonic series truncation |
| $J$ | truncation function for the meridional wavenumber |
| $\delta_{nm}$ | Kronecker delta |
| $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ | unit vector components corresponding to $(x, y, z)$ |
| $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ | two-dimensional velocity vector |
| $D = \nabla \cdot \mathbf{V}$ | horizontal divergence |
| $\Omega$ | angular speed of earth's rotation |
| $f = 2\Omega\sin\varphi$ | Coriolis parameter |
| $g$ | sea-level gravity, used with height in geopotential meters |
| $h$ | height of the shallow water layer |
| $\Phi = gh$ | geopotential energy of top of water layer |
| $\zeta$ | vorticity in the horizontal plane |
| $\Psi$ | the stream function (horizontal part of vector potential) |
| $\chi$ | the scalar potential |
| $U, V$ | modified velocity components |
| | Note: $V \neq \lvert\mathbf{V}\rvert$ |
| $\epsilon_n^m$ | factor used in derivatives of spherical harmonics |
| $Z_n^m, A_n, B_n, C_n, D_n, E_n$ | temporary Fourier coefficients |

There is still at least one unfound bug in this writeup, but the general flow of the math leading to the spectral solution for a GCM is correct. There are some equations where the references disagree with each other or with me. Conventions on when to include various factors of $a$ (earth radius) in the spherical harmonic formulations vary, as does the choice of including the renormalization factor $\xi_n^m$ within the associated Legendre function $P_n^m(\mu)$ or separating it as done here.

# Legendre Functions and Spherical Harmonics

Earth is spherical, and the spherical coordinate system used in meteorology is $(z, \varphi, \lambda)$ where

| | |
|---|---|
| $z \equiv r - a$ | altitude above sea level |
| $\varphi$ | latitude |
| $\lambda$ | longitude |

This varies from the standard mathematical coordinate system $(r, \theta, \varphi)$.

In cartesian coordinates, the Laplacian would be

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{1}$$

and we need to put that into the spherical coordinate system. The transform uses the differential of arc length:

$$d\mathbf{s} = \mathbf{i}\, a \, \cos\varphi \, d\lambda + \mathbf{j}\, a \, d\varphi \tag{2}$$

$$\nabla = \frac{\mathbf{i}}{a \, \cos\varphi} \frac{\partial}{\partial \lambda} \frac{\mathbf{j}}{a} \frac{\partial}{\partial \varphi} \tag{3}$$

$$\nabla^2 T = \frac{1}{a^2 \, \cos^2\varphi} \frac{\partial^2 T}{\partial \lambda^2} + \frac{1}{a^2 \, \cos\varphi} \frac{\partial}{\partial \varphi} \left( \cos\varphi \frac{\partial T}{\partial \varphi} \right) \tag{4}$$

Insert a useful coordinate transformation that gets rid of most references to trigonometric functions.

$$\mu \equiv \sin\varphi \tag{5}$$

$$d\mu = \cos\varphi \, d\varphi \tag{6}$$

$$d\varphi = \frac{d\mu}{\cos\varphi} = \frac{d\mu}{\sqrt{1-\mu^2}} \tag{7}$$

$$\cos\varphi = \sqrt{1-\mu^2} \tag{8}$$

$$\begin{aligned} \nabla^2 T &= \frac{1}{a^2 \, \cos^2\varphi} \frac{\partial^2 T}{\partial \lambda^2} + \frac{1}{a^2 \, \cos\varphi} \frac{\partial}{\partial \varphi} \left( \cos\varphi \frac{\partial T}{\partial \varphi} \right) \\ &= \frac{1}{a^2 (1-\mu^2)} \frac{\partial^2 T}{\partial \lambda^2} + \frac{\sqrt{1-\mu^2}}{a^2 \sqrt{1-\mu^2}} \frac{\partial}{\partial \mu} \left( (1-\mu^2) \frac{\partial T}{\partial \mu} \right) \\ &= \frac{1}{a^2 (1-\mu^2)} \frac{\partial^2 T}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial}{\partial \mu} \left( (1-\mu^2) \frac{\partial T}{\partial \mu} \right) \end{aligned} \tag{9}$$

**Diffusion on a sphere.**

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T == \frac{\kappa}{a^2} \left[ \frac{1}{1-\mu^2} \frac{\partial^2 T}{\partial \lambda^2} + \frac{\partial}{\partial \mu} \left( (1-\mu^2) \frac{\partial T}{\partial \mu} \right) \right] \tag{10}$$

Assume a separation of variables

$$T(\lambda, \mu, t) = P(\mu) \, Q(\lambda) \, S(t) \tag{11}$$

and plug these back into the diffusion PDE

$$PQ \frac{dS}{dt} = \frac{\kappa}{a^2} \left[ \frac{PS}{1-\mu^2} \frac{d^2 Q}{d\lambda^2} + QS \frac{d}{d\mu} \left( (1-\mu^2) \frac{dP}{d\mu} \right) \right] \tag{12}$$

in which note that $\partial$ has changed into $d$ because these are now all single-variable functions. Multiply by $a^2(1-\mu^2)/(\kappa PQS)$:

$$\frac{a^2(1-\mu^2)}{\kappa}\frac{1}{S}\frac{dS}{dt} = \frac{1}{Q}\frac{d^2Q}{d\lambda^2} + \frac{1-\mu^2}{P}\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} \tag{13}$$

Now the first term after the equals sign contains only terms in $\lambda$ so it can be separated out

$$\frac{1}{Q}\frac{d^2Q}{d\lambda^2} = \text{functions of only } (\mu, t) = \text{const} = -m^2 \tag{14}$$

where the separation constant, $-m^2$, as usual makes it obvious that someone has been reading ahead. This is the general harmonic equation, in which solutions will be constructed from

$$Q_m = e^{im\lambda} = \cos m\lambda + i\,\sin m\lambda \tag{15}$$

wherein $i = \sqrt{-1}$ puts us temporarily into complex numbers. (When these are used in meteorology, Fourier coefficients will always have sufficient factors of $i$ to assure that the solutions are real.) The qualitatively understood result is that solutions in longitude will consist of sums of periodic functions of various wavenumbers. The obvious fact that $Q_m(\lambda) = Q_m(\lambda + 2\pi)$ for all $m$ and all $\lambda$ means that $m$ will always be integers.

We could also quickly separate out the equation for $S(t)$ and obtain an exponential damping term, but that is not needed for the spectral model solution, we need the spatial eigenfunctions.

Having separated out and solved $Q(\lambda)$, the remaining parts of the equation are separable into

$$\frac{a^2}{\kappa}\frac{1}{S}\frac{dS}{dt} = \frac{1}{P}\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} - \frac{m^2}{1-\mu^2} \tag{16}$$

in which the right side contains all variation in terms of $\mu$. We invoke an apparently miraculous separation constant:

$$\frac{1}{P}\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} - \frac{m^2}{1-\mu^2} = -n(n+1) \tag{17}$$

This is called the Associated Legendre Differential Equation and it can be solved, but we will only do a simplified version as an example.

First phase: set $m = 0$ and assume *zonal symmetry* (as in Budyko-Sellers type zonal models) in which every point on a latitude circle has the same value, $Q(\lambda) = \text{const}$. The equation simplifies into the Legendre Differential Equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} + n(n+1)P = 0 \tag{18}$$

This can be solved by assuming a power-series solution.

$$P(\mu) = \sum_{j=0}^{\infty} a_j \mu^j \tag{19}$$

$$\frac{dP}{d\mu} = \sum_{j=1}^{\infty} a_j\, j\, \mu^{j-1} = \sum_{j=0}^{\infty} a_{j+1}(j+1)\mu^j \tag{20}$$

$$\frac{d^2P}{d\mu^2} = \sum_{j=2}^{\infty} a_j\, j(j-1)\,\mu^{j-2} = \sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)\mu^j \tag{21}$$

$$\mu^2\frac{dP}{d\mu} = \sum_{j=0}^{\infty} a_{j+1}(j+1)\mu^{j+2} \tag{22}$$

$$\frac{d}{d\mu}\left(\mu^2\frac{dP}{d\mu}\right) = \sum_{j=0}^{\infty} a_{j+1}(j+1)(j+2)\mu^{j+1} = \sum_{j=0}^{\infty} a_j\, j(j+1)\mu^j \tag{23}$$

Plug these sums back into the Legendre equation

$$0 = \frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu} + n(n+1)P \tag{24}$$

$$= \sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)\mu^j - \sum_{j=0}^{\infty} a_j\, j(j+1)\mu^j + n(n+1)\sum_{j=0}^{\infty} a_j\mu^j \tag{25}$$

$$= \sum_{j=0}^{\infty} \mu^j\left[a_{j+2}(j+1)(j+2) - a_j j(j+1) + n(n+1)a_j\right] \tag{26}$$

Since $\mu \in [-1,+1]$, the only way for this last equation to be true is if the element in the square brackets vanishes for all $j$, which creates a *recursion formula*

$$a_{j+2} = a_j\frac{j(j+1)-n(n+1)}{(j+1)(j+2)} \tag{27}$$

Implication of the $j+2$ recursion is that we will have a set of polynomials, each consisting of either entirely even-power terms in $\mu$ or entirely odd-power terms in $\mu$ because the coefficients jump by order two.

**Convergence** of the series $P(\mu) = \sum_{j=0}^{\infty} a_j\mu^j$ is furthermore a difficult requirement to fulfil at the poles, $\mu = \pm 1$. Particularly at the North Pole, $\mu = 1$ so $\mu^j = 1\ \forall j$ and we are dealing with the sum $P(1) = \sum_{j=0}^{\infty} a_j$. The recursion formula does not lead to a convergent sequence of $a_j$ values just by examining $j$,

$$\lim_{j\to\infty}\frac{a_{j+2}}{a_j} = \frac{O(j^2)}{O(j^2)} = 1 \tag{28}$$

with one special exception: if $n$ is assumed to be a positive integer, then the top of the recursion vanishes for $j = n$, and the series terminates.

We thus get two polynomial series, odd and even, each starting from a mathematically arbitrary $a_0$ or $a_1$. For the even series.

$$P(\mu) = \sum_{\substack{j=0 \\ n \text{ even}}}^{n} a_j \mu^j$$

$$\begin{aligned}
n &= 0: \\
&P_0 = a_0 \\
n &= 2: \\
&a_2 = a_0 \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} \\
&\quad = a_0 \frac{0 \cdot 1 - 2 \cdot 3}{1 \cdot 2} \\
&\quad = -3a_0 \\
&P_2 = a_0(1 - 3\mu^2) \\
n &= 4: \\
&a_2 = a_0 \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} \\
&\quad = a_0 \frac{0 \cdot 1 - 4 \cdot 5}{1 \cdot 2} \\
&\quad = -10a_0 \\
&a_4 = a_2 \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} \\
&\quad = a_2 \frac{2 \cdot 3 - 4 \cdot 5}{3 \cdot 4} \\
&\quad = -\frac{14}{12} a_2 = +\frac{140}{12} a_0 = \frac{35}{3} a_0 \\
&P_4 = a_0 \left(1 - 10\mu^2 + \frac{35}{3}\mu^4\right)
\end{aligned}$$

and so on. The value of $a_0$ may be arbitrarily defined for each different polynomial, but the final norm used to define the *Legendre Polynomials* is that $P_n(1) = 1$ for all $n$, so

$$P_0 = 1 \tag{29$a$}$$

$$P_2 = \frac{1}{2}(3\mu^2 - 1) \tag{29$b$}$$

$$P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3) \tag{29$c$}$$

and a set of odd Legendre polynomials can be defined using the same procedure.

$$P_1 = \mu \tag{30$a$}$$

$$P_3 = \frac{1}{2}(5\mu^3 - 3\mu) \tag{30$b$}$$

$$P_5 = \frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu) \tag{30$c$}$$

**Some Properties of the Legendre Polynomials.**

1) **Orthogonal**

$$\int_{-1}^{+1} P_n(\mu)\, P_m(\mu)\, d\mu = \frac{2}{2n+1}\delta_{nm} \tag{31}$$

where $\delta_{nm}$ is the *Kronecker delta*

$$\delta_{nm} \equiv \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \tag{32}$$

2) **Recursive**

$$(n+1)P_{n+1}(\mu) = (2n+1)\mu\, P_n(\mu) - n\, P_n(\mu) \tag{33}$$

(This is implemented for generating values of high-order Legendre Polynomials in Fortran subroutines available in the `cld` directory.)

3) **Range** $|P_n(\mu)| \leq 1\ \forall n,\ \mu \in [-1, +1]$

4) **Values at zero.** Two properties true of all sets of odd and even functions (compare sine and cosine, or odd and even powers of $x$, for example).

$$P_n(0) = 0 \qquad n \text{ odd} \tag{34}$$

$$\left.\frac{dP_n}{d\mu}\right)_0 = 0 \qquad n \text{ even} \tag{35}$$

5) **Normalization convention.** $P_n(1) = 1\ \forall n$. By the definitions of odd and even functions, this leads to $P_n(-1) = 1$ for even polynomials and $P_n(-1) = -1$ for odd polynomials.

6) **Integrals on range for even polynomials.**

$$\int_0^1 P_n(\mu)\, d\mu = 0\ \forall n > 0 \text{ (even)} \tag{36}$$

by symmetry then also,

$$\int_{-1}^0 P_n(\mu)\, d\mu = 0\ \forall n > 0 \text{ (even)}$$

(Note that $\int_{-1}^{+1} P_n(\mu)\, d\mu = 0\ \forall n > 0$, even or odd, where the even result follows from summing the previous two, and any odd functions whatsoever integrate to zero over any range symmetric about zero.)

7) **Roots.** Legendre Polynomial $P_n(\mu)$ has $n$ real roots in the range $[-1, +1]$. For even polynomials, the roots form a set that is symmetric about 0, and for odd polynomials, the roots include 0 plus a set that is symmetric about 0.

**Fourier Legendre Series.**

Any function of latitude, $f(\varphi)$, if sufficiently well-behaved that the integrals below exist, can be replaced by a Fourier-Legendre series, in the same manner that any periodic function on a cartesian space can be replaced with a Fourier trigonometric series. The proof is totally analogous to the Fourier sine or cosine series, and requires that we work in $\mu = \sin\varphi$ in the range $[-1, +1]$. Assert the existence of a set of coefficients $f_n$ such that

$$f(\mu) = \sum_{n=0}^{\infty} f_n P_n(\mu) \tag{37}$$

then do the reverse proof

$$\begin{aligned}
\int_{-1}^{+1} f(\mu)\, P_n(\mu)\, d\mu &= \int_{-1}^{+1} \left( \sum_{m=0}^{\infty} f_m P_m(\mu) \right) P_n(\mu)\, d\mu \\
&= \sum_{m=0}^{\infty} f_m \int_{-1}^{+1} P_m(\mu)\, P_n(\mu)\, d\mu \\
&= \sum_{m=0}^{\infty} f_m \frac{2}{2n+1} \delta_{nm} \\
&= f_n \frac{2}{2n+1}
\end{aligned} \tag{38}$$

so

$$f_n = \frac{2n+1}{2} \int_{-1}^{+1} f(\mu)\, P_n(\mu)\, d\mu \tag{39}$$

A useful note about the first coefficient, for which $P_0 = 1$:

$$a_0 = \frac{1}{2} \int_{-1}^{+1} f(\mu)\, d\mu = \bar{f} \tag{40}$$

**Associated Legendre Functions and Spherical Harmonics.**

Returning to the Associated Legendre Differential Equation, before the assumption of $m = 0$,

$$\frac{1}{P} \frac{d}{d\mu} (1 - \mu^2) \frac{dP}{d\mu} - \frac{m^2}{1 - \mu^2} = -n(n+1) \tag{17}$$

solutions can be obtained for series that require truncation via both $m$ (which has already been required to be an integer) and $n$. The actual derivation of the eigenfunctions is well beyond our interest, but what we obtain is a set of *Associated Legendre Functions* $P_n^m(\mu)$ in which $n$ is still a nonnegative integer and $m$ is an integer in the range $[-n, +n]$. The generating equation for these is

$$P_n^m(\mu) = \frac{(-1)^m}{2^n n!} (1 - \mu^2)^{m/2} \frac{d^{(m)}}{d\mu^{(m)}} P_n(\mu) \tag{41}$$

leading to just a few examples

$$P_n^0 = \frac{1}{2^n n!} P_n(\mu) \tag{42$a$}$$

$$P_1^0 = \frac{1}{2}\mu = \frac{1}{2} \sin phi \tag{43$b$}$$

$$P_1^1 = -\frac{1}{2} (1 - \mu^2)^{1/2} = -\frac{1}{2} \cos\varphi \tag{44$c$}$$

in which the beginning of one trend is barely noticeable: the oddness or evenness of an Associated Legendre function is based on the value of $n - |m|$, and that value is also the number of roots in the range $(-1, +1)$. An additional characteristic is that the Associated Legendre functions that are even in $n - |m|$ are 0 at $\pm 1$.

We combine Associated Legendre functions with the longitudinal terms to get a set of *Spherical Harmonics*.

$$Y_n^m(\mu, \lambda) = \xi_n^m P_n^m(\mu) e^{im\lambda} \tag{45}$$

where $\xi_n^m$ is a new normalization factor

$$\xi_n^m = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \tag{46}$$

The spherical harmonics are orthogonal in two dimensions

$$\int_0^{2\pi} d\lambda \int_{-1}^{+1} d\mu Y_n^{m*} Y_k^l = \delta_{ml} \delta_{nk} \tag{47}$$

where $Y_n^{m*}$ is the complex conjugate of $Y_n^m$. These spherical harmonics each have a pattern of longitude-line roots or latitude-line roots, in which $Y_n^0$ will be *zonal harmonics* consisting of factors of the Legendre polynomials, $Y_n^n$ will be *sectoral harmonics* that consist of factors of wave patterns around the globe, and any others are called *tesseral harmonics*.

Any map field $A(\varphi, \lambda) = A(\mu, \lambda)$ that is mathemtically well-behaved can be represented as Fourier spherical harmonic series

$$A(\mu, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} A_n^m Y_n^m(\mu, \lambda) \tag{48}$$

where the $A_n^m$ coefficients can be found by integration, using the orthogonality of the spherical harmonics,

$$A_n^m = \int_0^{2\pi} d\lambda \int_{-1}^{+1} d\mu A(\mu, \lambda) Y_n^{m*}(\mu, \lambda) \tag{49}$$

**Truncation.** In practice, no Fourier series is ever summed to infinity. The usual form for a truncation is

$$A(\mu, \lambda) \doteq \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} A_n^m Y_n^m(\mu, \lambda) \tag{50}$$

where $M$ is the truncation zonal wavenumber (i.e., the 15 in "R15" or the 42 in "T42."), $J(m)$ is the truncation function applied to the meridional wavenumber parameter. The two most common forms of $J$ are

$$J(m) = \begin{cases} M & \text{Triangular truncation} \\ M + m & \text{Rhomboidal truncation} \end{cases}$$

# Shallow Water Equations

**Momentum equations** in two dimensions

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0 \tag{51}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0 \tag{52}$$

and in vector form

$$\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V}\cdot\nabla)\mathbf{V} + f\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0 \tag{53}$$

reminder:

$$\mathbf{k}\times\mathbf{V} = \mathbf{k}\times(u\mathbf{i}+v\mathbf{j}) = u\mathbf{j} - v\mathbf{i}$$

Take an apparently backwards step by bringing in **vorticity**

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{54}$$

and kinetic energy per unit mass

$$\frac{1}{2}\mathbf{V}\cdot\mathbf{V} = \frac{u^2+v^2}{2} \tag{55}$$

$$\nabla\left(\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) = \mathbf{i}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) + \mathbf{j}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)$$

Add and subtract $v(\partial u/\partial y)$ in first ($\mathbf{i}$) term and $u(\partial v/\partial x)$ in second term. Rearrange terms.

$$\begin{aligned}
\nabla\left(\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) &= \mathbf{i}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} - v\frac{\partial u}{\partial y}\right) \\
&\quad + \mathbf{j}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} - u\frac{\partial v}{\partial x}\right) \\
&= \mathbf{i}\left[\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \\
&\quad + \mathbf{j}\left[\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
&= \mathbf{i}[(\mathbf{V}\cdot\nabla)u - v\zeta] + \mathbf{j}[(V\cdot\nabla)v - u\zeta] \\
&= (\mathbf{V}\cdot\nabla)\mathbf{V} + \zeta\mathbf{k}\times\mathbf{V}
\end{aligned}$$

Plug this result back into the vector momentum equation (53):

$$\frac{\partial \mathbf{V}}{\partial t} + (f+\zeta)\mathbf{k}\times\mathbf{V} + \nabla\left(\Phi + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) = 0 \tag{56}$$

**Mass continuity**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{57}$$

Integrate (57) over the depth of the layer, assume horizontal velocities do not vary with height.

$$\begin{aligned}
0 &= \int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz \\
&= h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(h) - w(0)
\end{aligned}$$

Use $w(0) = 0$ and $w(h) = dh/dt$, expand the total derivative for $h$ into the local and advected components:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{58}$$

Further simplify this by assuming that $h$ is subject to small variations around a larger mean state, such that if

$$h = \bar{h} + h'$$

where $\bar{h}$ is a constant mean height and $h'$ is the perturbation around that mean, then (58) can be rewritten as

$$\frac{\partial h'}{\partial t} + u\frac{\partial h'}{\partial x} + v\frac{\partial h'}{\partial y} + \bar{h}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

in which the derivatives get $h'$ since the constant term has no derivative, and the integrated term gets $\bar{h}$ since $\bar{h} \gg h'$. Extend $\Phi$ in the same way

$$\Phi = \bar{\Phi} + \Phi'; \qquad \Phi' = gh'; \qquad \bar{\Phi} = g\bar{h}$$

Then mass continuity becomes

$$\begin{aligned} \frac{\partial \Phi'}{\partial t} &= -(\mathbf{V}\cdot\nabla)\Phi' - \bar{\Phi}\nabla\cdot\mathbf{V} \\ &= -(\mathbf{V}\cdot\nabla)\Phi' - \bar{\Phi}D \end{aligned} \tag{59}$$

where $D = \nabla\cdot\mathbf{V}$ is the horizontal divergence.

**Helmholtz's Theorem** lets us break any "reasonable" (i.e., continuous and sufficiently differentiable) vector field into an irrotational component (a vector field for which the vorticity vanishes) and a nondivergent component (a vector field for which the divergence vanishes). This allows us to create two scalar fields, a **stream function** $\Psi$ and a **scalar potential** $\chi$, such that

$$\mathbf{V} = \mathbf{k}\times\nabla\Psi + \nabla\chi \tag{60}$$

where $\nabla\chi$ is the irrotational component (because $\nabla\times\nabla\chi = 0$ for *any* scalar field) and $\mathbf{k}\times\nabla\Psi$ is nondivergent, because

$$\begin{aligned} \nabla\cdot(\mathbf{k}\times\nabla\Psi) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)\cdot\left(\mathbf{i}\frac{\partial\Psi}{\partial y} - \mathbf{j}\frac{\partial\Psi}{\partial x}\right) \\ &= \frac{\partial^2\Psi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y\partial x} \\ &= 0 \end{aligned}$$

The interchange in the order of differentiation is always possible for well-behaved physical fields. Then:

$$\begin{aligned} \zeta &= \mathbf{k}\cdot\nabla\times\mathbf{V} \\ &= \mathbf{k}\cdot[\nabla\times(\mathbf{k}\times\nabla\Psi) + \nabla\times\nabla\chi] \\ &\qquad \text{(expand)} \qquad\qquad \text{(vanishes)} \\ &= \mathbf{k}\cdot[\mathbf{k}(\nabla\cdot\nabla\Psi) - \nabla\Psi(\nabla\cdot\mathbf{k}) - (\nabla\Psi\cdot\nabla)\mathbf{k} - (\mathbf{k}\cdot\nabla)\nabla\Psi] \\ &\qquad \text{(vertical derivatives and derivatives of } \mathbf{k} \text{ vanish)} \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{k}\cdot[\mathbf{k}\nabla\cdot\nabla\Psi] \\
&= \nabla^2\Psi \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
D &= \nabla\cdot\mathbf{V} \\
&= \nabla\cdot(\mathbf{k}\times\nabla\Psi) + \nabla\cdot\nabla\chi \\
&= \nabla\Psi\cdot(\nabla\times\mathbf{k}) - \mathbf{k}(\nabla\times\nabla\Psi) + \nabla^2\chi \\
&= \nabla^2\chi \qquad (62)
\end{aligned}
$$

**The Divergence Equation.** Create the divergence equation by taking the divergence of the vector form of the momentum equation (6). The second step requires the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$
\begin{aligned}
\nabla\cdot\frac{\partial\mathbf{V}}{\partial t} &= -\nabla\cdot[(f+\zeta)\mathbf{k}\times\mathbf{V}] - \nabla^2\left(\Phi' + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) \\
\frac{\partial D}{\partial t} &= -\mathbf{k}\cdot[\nabla\times(f+\zeta)\mathbf{V}] - (f+\zeta)\mathbf{V}(\nabla\times\mathbf{k}) - \nabla^2\left(\Phi' + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) \\
&= -\mathbf{k}\cdot[\nabla\times(f+\zeta)\mathbf{V}] - \nabla^2\left(\Phi' + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) \qquad (63)
\end{aligned}
$$

**The Vorticity Equation** is created by taking the vertical component of the curl of the momentum equation (6). [$\mathbf{k}\cdot\nabla\times$ (momentum equation)].

$$
\mathbf{k}\cdot\nabla\times\frac{\partial\mathbf{V}}{\partial t} = -\mathbf{k}\cdot\nabla\times[\mathbf{k}\times(f+\zeta)\mathbf{V}] + \underbrace{\mathbf{k}\cdot\nabla\times\nabla\left(\Phi' + \frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right)}_{\text{(vanishes)}}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}(\mathbf{k}\cdot\nabla\times\mathbf{V}) &= -\mathbf{k}\cdot\{\mathbf{k}[\nabla\cdot(f+\zeta)\mathbf{V}] - [(f+\zeta)\mathbf{V}]\nabla\cdot\mathbf{k} \\
&\qquad + [(f+\zeta)\mathbf{V}\cdot\nabla]\mathbf{k} - (\mathbf{k}\cdot\nabla)[(f+\zeta)\mathbf{V}] \\
\frac{\partial\zeta}{\partial t} &= -\nabla\cdot[(f+\zeta)\mathbf{V}] \qquad (64)
\end{aligned}
$$

**Spherical coordinate modifications.** Define a pair of modified velocity components

$$U \equiv u\cos\varphi; \qquad V \equiv v\cos\varphi \qquad (65)$$

then

$$\mathbf{V} = \mathbf{i}\frac{U}{\cos\varphi} + \mathbf{j}\frac{V}{\cos\varphi} \qquad (66)$$

We will also occasionally need forms of gradient, divergence, and curl in meteorological spherical coordinates. Note that these differ from math textbooks slightly because we use $\varphi$ as latitude while the standard mathematical spherical coordinate system uses colatitude, which starts with 0 at the North Pole.

$$
\begin{aligned}
\nabla\Phi &= \frac{\mathbf{i}}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + \frac{\mathbf{j}}{a}\frac{\partial\Phi}{\partial\varphi} \\
\nabla\cdot\mathbf{V} &= \frac{1}{a\cos\varphi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
\mathbf{k}\cdot(\nabla\times\mathbf{V}) &= \frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right]
\end{aligned}
$$

Use these to construct spherical-coordinate forms of the vorticity and divergence equations.

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^2\Psi &= -\nabla\cdot[(\nabla^2\Psi+f)\mathbf{V}] \\ &= -\nabla\cdot(f\mathbf{V})-\nabla\cdot(\mathbf{V}\nabla^2\Psi) \\ &= -\nabla f\cdot\mathbf{V}f\nabla\cdot\mathbf{V}-\nabla\cdot(\mathbf{V}\nabla^2\Psi) \qquad (67)\\ \nabla f &= \frac{\mathbf{i}}{a\cos\varphi}\frac{\partial f}{\partial\lambda}+\frac{\mathbf{j}}{a}\frac{\partial f}{\partial\varphi}=\frac{\mathbf{j}}{a}2\Omega\cos\varphi \\ \nabla f\cdot\mathbf{V} &= \frac{V}{\cos\varphi}\frac{2\Omega\cos\varphi}{a}=\frac{2\Omega V}{a} \qquad (68a)\\ f\nabla\cdot\mathbf{V} &= f\nabla^2\chi = 2\Omega\sin\varphi\nabla^2\chi \qquad (68b)\\ \nabla\cdot(V\nabla^2\Psi) &= \nabla\cdot\left(\mathbf{i}\frac{U\nabla^2\Psi}{\cos\varphi}+\mathbf{j}\frac{V\nabla^2\Psi}{\cos\varphi}\right) \\ &= \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\lambda}(U\nabla^2\Psi)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{V\nabla^2\Psi}{\cos\varphi}\right) \\ &= \frac{1}{a\cos^2\varphi}\left[\frac{\partial(U\nabla^2\Psi)}{\partial\lambda}+\cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial\varphi}\right] \qquad (68c)\end{aligned}$$

Put (68)s back into (67):

$$\begin{aligned}\frac{\partial(\nabla^2\Psi)}{\partial t} &= -\frac{1}{a\cos^2\varphi}\left[\frac{\partial(U\nabla^2\Psi)}{\partial\lambda}+\cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial\varphi}\right] \\ &\quad -2\Omega\left(\frac{\mathbf{V}}{a}+(\nabla^2\chi)\sin\varphi\right) \qquad (69)\end{aligned}$$

Now do the same treatment for the divergence equation.

$$\frac{\partial\nabla^2\chi}{\partial t} = -\mathbf{k}\cdot[\nabla\times(f+\zeta)\mathbf{V}]-\nabla^2\left(\Phi'+\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) \qquad (70)$$

Last term is easiest, deal with it first.

$$\nabla^2\left(\Phi'+\frac{1}{2}\mathbf{V}\cdot\mathbf{V}\right) = \nabla^2\left(\Phi'+\frac{U^2+V^2}{2\cos^2\varphi}\right) \qquad (71a)$$

In the first term on the right side of (70), deal with $f$ separately from $\zeta$. Easiest first again, so for $f$:

$$\begin{aligned}\mathbf{k}\cdot\nabla\times(f\mathbf{V}) &= \mathbf{k}\cdot(\nabla f\times\mathbf{V}+f\nabla\times\mathbf{V}) \\ &= \mathbf{k}\cdot(\nabla f\times\mathbf{V})+f\nabla^2\Psi \\ &= \mathbf{k}\cdot\left[\left(\mathbf{j}\frac{2\Omega\cos\varphi}{a}\right)\times\left(\mathbf{i}\frac{U}{\cos\varphi}+\mathbf{j}\frac{V}{\cos\varphi}\right)\right]+f\nabla^2\Psi \\ &= -\frac{2\Omega U}{a}+f\nabla^2\Psi \qquad (71b)\end{aligned}$$

Now deal with $\zeta$

$$\begin{aligned}\mathbf{k}\cdot\nabla\times(\zeta\mathbf{V}) &= \mathbf{k}\cdot\nabla\times\left(\mathbf{i}\frac{U\nabla^2\Psi}{\cos\varphi}+\mathbf{j}\frac{\mathbf{V}\nabla^2\Psi}{\cos\varphi}\right) \\ &= \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{U\nabla^2\Psi}{\cos\varphi}\right)-\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{V\nabla^2\Psi}{\cos\varphi}\right)\right] \\ &= \frac{1}{a\cos^2\varphi}\left(\frac{\partial(U\nabla^2\Psi)}{\partial\lambda}-\cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial\varphi}\right) \qquad (71c)\end{aligned}$$

Put (71)s back into (70).

$$\begin{aligned}\frac{\partial(\nabla^2\chi)}{\partial t} = \frac{-1}{a\cos^2\varphi}\left(\frac{\partial(U\nabla^2\Psi)}{\partial\lambda} - \cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial\varphi}\right) \\ + \frac{2\Omega U}{a} + f\nabla^2\Psi - \nabla^2\left(\Phi' + \frac{U^2+V^2}{2\cos^2\varphi}\right)\end{aligned} \tag{72}$$

Similar treatment for the continuity equation.

$$\begin{aligned}\frac{\partial\Phi'}{\partial t} &= -\nabla\cdot(\Phi'\mathbf{V}) - \bar{\Phi}D \\ &= -\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\Phi' U}{\cos\varphi}\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\Phi' V}{\cos\varphi}\right)\right] - \bar{\Phi}\nabla^2\chi \\ &= \frac{-1}{a\cos^2\varphi}\left(\frac{\partial(\Phi' U)}{\partial\lambda} + \cos\varphi\frac{\partial(\Phi' V)}{\partial\varphi}\right) - \bar{\Phi}\nabla^2\chi\end{aligned} \tag{73}$$

Now have three main prognostic equations, (69), (72), and (73), which are prognostic for $\nabla^2\Psi$, $\nabla^2\chi$, and $\Phi'$, respectively. To close the system, we need diagnostic relations for $U$ and $V$ in terms of the prognostic variables.

$$\begin{aligned}U &= u\cos\varphi \\ &= \mathbf{i}\cdot(\mathbf{k}\times\nabla\Psi + \nabla\chi)\cos\varphi \\ &= \left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi} + \frac{1}{a\cos\varphi}\frac{\partial\chi}{\partial\lambda}\right)\cos\varphi \\ &= \frac{-\cos\varphi}{a}\frac{\partial\Psi}{\partial\varphi} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda}\end{aligned} \tag{74a}$$

$$\begin{aligned}V &= v\cos\varphi \\ &= \mathbf{j}\cdot(\mathbf{k}\times\nabla\Psi + \nabla\chi)\cos\varphi \\ &= \left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial\chi}{\partial\varphi}\right)\cos\varphi \\ &= \frac{1}{a}\frac{\partial\Psi}{\partial\lambda} + \frac{\cos\varphi}{a}\frac{\partial\chi}{\partial\varphi}\end{aligned} \tag{74b}$$

# Spherical Harmonic Transform Method

**Spherical Harmonic truncated Fourier series.** Set up spherical harmonic Fourier series representations for each of the major fields.

$$\begin{aligned}\Psi &= a^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Psi_n^m Y_n^m \\ &\doteq a^2 \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} \Psi_n^m Y_n^m \end{aligned} \tag{75}$$

where $M$ is the truncation zonal wavenumber, $J(m)$ is the truncation function applied to the meridional wavenumber parameter and the $\Psi_n^m$ are the spherical harmonic Fourier coefficients of the field $\Psi$. The Fourier coefficients are fit via the double integral

$$\Psi_n^m = \int_{-1}^{+1} d\mu \int_0^{2\pi} d\lambda \, \Psi(\mu, \lambda) Y{*}_n^m(\mu, \lambda) \tag{76}$$

where $Y{*}_n^m(\mu, \lambda))$ is the *complex conjugate* of $Y_n^m$,

$$Y{*}_n^m(\mu, \lambda) = \xi_n^m P_n^m(\mu) e^{-im\lambda} = \xi_n^m P_n^m(\mu)[\cos im\lambda - i \sin im\lambda]$$

(recall that $\cos -a = \cos a$ because cosine is an *even function.*) (76) works because the the spherical harmonics are orthogonal—orthogonality in complex eigenfunctions always involves the complex conjugate, viz

$$\int_{-1}^{+1} d\mu \int_0^{2\pi} d\lambda \, Y_n^m(\mu, \lambda) Y{*}_k^j(\mu, \lambda) = \delta_{nk}\delta_{mj} \tag{77}$$

Define sets of Fourier coefficients for each of the major fields, using (76) to define the coefficients. (We have deferred discussion of *how* to obtain these integrals efficiently for a numerically specified field on the globe.)

$$\chi \doteq a^2 \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} \chi_n^m Y_n^m \tag{78}$$

$$\Phi' \doteq a^2 \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} \Phi_n^m Y_n^m \tag{79}$$

$$U \doteq a \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} U_n^m Y_n^m \tag{80}$$

$$V \doteq a \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} V_n^m Y_n^m \tag{81}$$

**Derivatives of Spherical Harmonics.** The following formulae can be used for getting derivatives of fields from their Fourier coefficients.

$$(\mu^2 - 1)\frac{\partial Y_n^m}{\partial \mu} = n\sqrt{\frac{(n+1)^2 - m^2}{4(n+1)^2 - 1}} Y_{n+1}^m - (n+1)\sqrt{\frac{(n-1)^2 - m^2}{4(n-1)^2 - 1}} Y_{n-1}^m$$

Simplifying definition:

$$\epsilon_n^m \equiv \sqrt{\frac{n^2 - m^2}{4n^2 - 1}}$$

allows

$$(\mu^2 - 1)\frac{\partial Y_n^m}{\partial \mu} = n\epsilon_{n+1}^m Y_{n+1}^m - (n+1)\epsilon_{n-1}^m Y_{n-1}^m \tag{82}$$

The other derivative is easier:

$$\begin{aligned}\frac{\partial Y_n^m}{\partial \lambda} &= \xi_n^m P_n^m(\mu)\frac{\partial}{\partial \lambda}e^{im\lambda} \\ &= \xi_n^m P_n^m(\mu) im e^{im\lambda} \\ &= imY_n^m \end{aligned} \tag{83}$$

The most useful aspect of the spherical harmonics is that they are eigenfunctions of the Poisson equation for a sphere. They each satisfy the following equation

$$\nabla^2 Y_n^m + \frac{n(n+1)}{a^2} Y_n^m = 0 \tag{84}$$

**Removing the spatial derivatives.** Our three prognostic equations are the vorticity, divergence, and continuity equations, and we have two equations for the $U$ and $V$ components of velocity as well. Our purpose is to plug truncated spherical harmonic series into these equations in various places, and then use formulas for the derivatives of spherical harmonics to get rid of all spatial derivatives. By the time we are done, we want every term of these equations to be multiplying the same spherical harmonic (i.e., the same $n$ and $m$) so that we can make some use of orthogonality to simplify things. The process is straightforward at first, but has several twists and turns.

Along the way, we need some Fourier coefficients that are transformed on longitude only using sines and cosines, not on the whole sphere. Hence, $A_m, B_m, C_m, D_m$, and $E_m$ defined here are all functions of latitude ($\mu$).

$$U\nabla^2\Psi = a\sum_{m=-M}^{M} A_m e^{im\lambda} \qquad V\nabla^2\Psi = a\sum_{m=-M}^{M} B_m e^{im\lambda} \tag{85}$$

$$U\Phi' = a^3\sum_{m=-M}^{M} C_m e^{im\lambda} \qquad V\Phi' = a^3\sum_{m=-M}^{M} D_m e^{im\lambda} \tag{86}$$

$$\frac{U^2+V^2}{2}\nabla^2\Psi = a^2\sum_{m=-M}^{M} E_m e^{im\lambda} \tag{87}$$

and these coefficients are calculated from an inverse Fourier transform, as in

$$A_m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} d\lambda\, U\nabla^2\Psi e^{-im\lambda} \tag{88}$$

Proceeding with expansions for velocity components:

$$\begin{aligned} U &= \frac{-\cos\varphi}{a}\frac{\partial \Psi}{\partial \varphi} + \frac{1}{a}\frac{\partial \chi}{\partial \lambda} \qquad (24a) \\ &= \frac{(1-\mu^2)^{\frac{1}{2}}}{a}\frac{\partial \Psi}{\partial \mu}\frac{d\mu}{d\varphi} + \frac{1}{a}\frac{\partial \chi}{\partial \lambda} \end{aligned}$$

$$\left[\textbf{Note:} \frac{d\mu}{d\varphi} = \frac{d(\sin\varphi)}{d\varphi} = \cos\varphi = \sqrt{1-\mu^2}\right]$$

$$
\begin{aligned}
&= \frac{1-\mu^2}{a}\frac{\partial \Psi}{\partial \mu}\frac{d\mu}{d\varphi} + \frac{1}{a}\frac{\partial \chi}{\partial \lambda} \\
&= a(1-\mu^2)\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \Psi_n^m \frac{\partial Y_n^m}{\partial \mu} + a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \chi_n^m \frac{\partial Y_n^m}{\partial \lambda} \\
&= a(1-\mu^2)\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \Psi_n^m \left(\frac{1}{\mu^2-1}[n\epsilon_{n+1}^m Y_{n+1}^m - (n+1)\epsilon_n^m Y_{n-1}^m]\right) \\
&\quad + a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \chi_n^m im Y_n^m \\
&= a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} Y_n^m[(n-1)\epsilon_n^m \Psi_{n-1}^m - (n+2)\epsilon_{n+1}^m \Psi_{n+1}^m + im\chi_n^m]
\end{aligned}
$$

Since the quantity in square brackets corresponds to the position of $U_n^m$ in the truncated Fourier representation of $U$, we have

$$U_n^m = (n-1)\epsilon_n^m \Psi_{n-1}^m - (n+2)\epsilon_{n+1}^m \Psi_{n+1}^m + im\chi_n^m \tag{89}$$

Similarly for $V$

$$
\begin{aligned}
V &= \frac{1}{a}\frac{\partial \Psi}{\partial \lambda} + \frac{\cos\varphi}{a}\frac{\partial \chi}{\partial \varphi} \qquad (24b) \\
&= \frac{1}{a}\frac{\partial \Psi}{\partial \lambda} + \frac{1-\mu^2}{a}\frac{\partial \chi}{\partial \mu} \\
&= a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \Psi_n^m im Y_n^m \\
&\quad + a(1-\mu^2)\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} \chi_n^m \left(\frac{1}{\mu^2-1}[n\epsilon_{n+1}^m Y_{n+1}^m - (n+1)\epsilon_n^m Y_{n-1}^m]\right) \\
&= a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} Y_n^m[im\Psi_n^m - n\epsilon_{n+1}^m \chi_{n+1}^m + (n+1)\epsilon_n^m \chi_{n-1}^m]
\end{aligned}
$$

and hence

$$V_n^m = im\Psi_n^m - n\epsilon_{n+1}^m \chi_{n+1}^m + (n+1)\epsilon_n^m \chi_{n-1}^m \tag{90}$$

We now move on to the vorticity equation, and things start to get messy.

$$
\begin{aligned}
\frac{\partial(\nabla^2\Psi)}{\partial t} &= -\frac{1}{a\cos^2\varphi}\left[\frac{\partial(U\nabla^2\Psi)}{\partial \lambda} + \cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial \varphi}\right] \\
&\quad - 2\Omega\left(\frac{V}{a} + (\nabla^2\chi)\sin\varphi\right) \qquad (19) \\
&= \frac{-1}{a(1-\mu^2)}\left[\frac{\partial(U\nabla^2\Psi)}{\partial \lambda} + (1-\mu^2)\frac{\partial(V\nabla^2\Psi)}{\partial \mu}\right] \\
&\quad - 2\Omega\left(\frac{V}{a} + \mu\nabla^2\chi\right) \qquad (91)
\end{aligned}
$$

Take (91) apart one piece at a time. Start with the left side.

$$\begin{aligned}
\frac{\partial(\nabla^2\Psi)}{\partial t} &= \frac{\partial}{\partial t}\nabla^2\left(a^2\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)}\Psi_n^m Y_n^m\right) \\
&= a^2\frac{\partial}{\partial t}\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)}\Psi_n^m\nabla^2 Y_n^m \\
&= -\frac{\partial}{\partial t}\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)}\Psi_n^m n(n+1)Y_n^m \\
&= -\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} n(n+1)Y_n^m\frac{\partial\Psi_n^m}{\partial t} \qquad (92)
\end{aligned}$$

Now the big term on the right side of (91). Define $Z$ as that term, and create a temporary spherical harmonic Fourier coefficient $Z_n^m$ to represent it, i.e.,

$$\begin{aligned}
Z(\mu,\lambda) &\equiv -\frac{1}{a\cos^2\varphi}\left(\frac{\partial(U\nabla^2\Psi)}{\partial\lambda} + (1-\mu^2)\frac{\partial(V\nabla^2\Psi)}{\partial\mu}\right) \qquad (93)a \\
&= a\sum_{m=-M}^{M}\sum_{n=|m|}^{J(m)} Z_n^m Y_n^m \qquad (93)b
\end{aligned}$$

then $Z_n^m$ must be calculated via the usual integrals. There are derivative-order interchanges and an integration by parts between the 2nd and 3rd equations:

$$\begin{aligned}
Z_n^m &= \int_{-1}^{+1}d\mu\int_0^{2\pi}d\lambda\, Z(\mu,\lambda)Y{^*}_n^m(\mu,\lambda) \\
&= \int_{-1}^{+1}d\mu\int_0^{2\pi}d\lambda\left\{\frac{1}{a(1-\mu^2)}\left(\frac{\partial(U\nabla^2\Psi)}{\partial\lambda} + \cos\varphi\frac{\partial(V\nabla^2\Psi)}{\partial\varphi}\right)\xi_n^m P_n^m e^{-im\lambda}\right\} \\
&= \frac{\xi_n^m}{a}\int_{-1}^{+1}d\mu\left\{\frac{P_n^m(\mu)}{(1-\mu^2)}\left(\int_0^{2\pi}d\lambda\, e^{-im\lambda}\frac{\partial}{\partial\lambda}\sum_{j=-M}^{M}A_j(\mu)e^{ij\lambda}\right)\right. \\
&\qquad \left. + \int_0^{2\pi}d\lambda\, e^{-im\lambda}\frac{\partial}{\partial\mu}(U\nabla^2\Psi P_n^m) - \int_0^{2\pi}d\lambda\, e^{-im\lambda}(V\nabla^2\Psi)\frac{dP_n^m}{d\mu}\right\} \qquad (94)
\end{aligned}$$

We can get rid of the 2nd double integral of (94) using Leibniz's theorem for moving a derivative outside an integral:

$$\int_a^b\frac{\partial f(x,t)}{\partial x}dt = \frac{d}{dx}\int_a^b f(x,t)dt$$

from which we make the double-integral extension

$$\begin{aligned}
\int_p^q dx\int_a^b\frac{\partial f(x,t)}{\partial x}dt &= \int_p^q dx\frac{d}{dx}\int_a^b f(x,t)dt \\
&= \int_p^q d\left(\int_a^b f(x,t)dt\right) \\
&= \int_a^b f(q,t)dt - \int_a^b f(p,t)dt
\end{aligned}$$

Taking the entire second double integral of (94) out,

$$\begin{aligned}\int_{-1}^{+1} d\mu \int_0^{2\pi} d\lambda\, e^{-im\lambda} \frac{\partial}{\partial\mu}(U\nabla^2\Psi P_n^m) &= \int_{-1}^{+1} d\mu\, \frac{d}{d\mu}\left(\int_0^{2\pi} d\lambda\, e^{-im\lambda} U\nabla^2\Psi P_n^m\right) \\ &= \int_{\mu=-1}^{\mu=1} d\big(P_n^m(\mu)B_m(\mu)\big) \\ &= P_n^m(1)B_m(1) - P_n^m(-1)B_m(-1) \\ &= 0\end{aligned}$$

because $B_m$ are the coefficients for $V$, which must vanish at the poles.

Resuming (94) with the second integral removed, we will need to know that $e^{im\lambda}$ has the following orthogonality property:

$$\int_0^{2\pi} d\lambda\, e^{-im\lambda} e^{ij\lambda} = 2\pi\delta_{jm} = \begin{cases} 2\pi & j = m \\ 0 & j \neq m \end{cases}$$

then

$$\begin{aligned} Z_n^m &= \frac{\xi_n^m}{a} \int_{-1}^{+1} d\mu \left\{ \frac{P_n^m(\mu)}{(1-\mu^2)} \left( \int_0^{2\pi} d\lambda\, e^{-im\lambda} \frac{\partial}{\partial\lambda} \sum_{j=-M}^{M} A_j(\mu) e^{ij\lambda} \right) \right. \\ &\qquad \left. - \int_0^{2\pi} d\lambda\, e^{-im\lambda} (V\nabla^2\Psi) \frac{dP_n^m}{d\mu} \right\} \\ &= \frac{\xi_n^m}{a} \int_{-1}^{+1} d\mu \left\{ \frac{P_n^m(\mu)}{(1-\mu^2)} 2\pi i j A_j \delta_{jm} - 2\pi B_m \frac{dP_n^m}{d\mu} \right\} \\ &= \frac{2\pi\xi_n^m}{a} \int_{-1}^{+1} d\mu \left( \frac{imA_m P_n^m(\mu)}{(1-\mu^2)} - 2\pi B_m \frac{dP_n^m}{d\mu} \right) \end{aligned} \tag{95}$$

The remaining terms of (91) are simpler.

$$-2\Omega\frac{V}{a} = -2\Omega \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} V_n^m Y_n^m \tag{96}$$

$$-2\Omega\mu\nabla^2\chi = 2\Omega \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} Y_n^m [n(n-1)\epsilon_n^m \chi_{n-1}^m + (n+1)(n+2)\epsilon_{n+1}^m \chi_{n+1}^m] \tag{97}$$

where (97) results from (84) and some manipulation of the Laplacian term.

Now reconstruct (91) with the intermediate results of (92), (95), (96), and (97).

$$\frac{\partial(\nabla^2\Psi)}{\partial t} = \frac{-1}{a(1-\mu^2)} \left[ \frac{\partial(U\nabla^2\Psi)}{\partial\lambda} + (1-\mu^2)\frac{\partial(V\nabla^2\Psi)}{\partial\mu} \right] - 2\Omega\left(\frac{V}{a} + \mu\nabla^2\chi\right) \tag{91}$$

$$\begin{aligned} &- a \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} n(n+1) Y_n^m \frac{\partial\Psi_n^m}{\partial t} \\ &= a \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} Y_n^m \left[ \frac{2\pi\xi_n^m}{a} \int_{-1}^{+1} d\mu \left( \frac{imA_m P_n^m(\mu)}{(1-\mu^2)} - 2\pi B_m \frac{dP_n^m}{d\mu} \right) \right] \quad (98) \\ &+ 2\Omega \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} Y_n^m [n(n-1)\epsilon_n^m \chi_{n-1}^m + (n+1)(n+2)\epsilon_{n+1}^m \chi_{n+1}^m - V_n^m] \end{aligned}$$

A result of the orthogonality of the $Y_n^m$ is that they are linearly independent, meaning that if

$$\sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} \alpha_n^m Y_n^m = \sum_{m=-M}^{M} \sum_{n=|m|}^{J(m)} \beta_n^m Y_n^m$$

then

$$\alpha_n^m = \beta_n^m, \ \forall m, n$$

This allows us to drop the summations and the spherical harmonics from (98) and equate the Fourier coefficients.

$$\begin{aligned} -n(n+1)\frac{\partial \Psi_n^m}{\partial t} = & \left[\frac{2\pi\xi_n^m}{a}\int_{-1}^{+1} d\mu \left(\frac{imA_m P_n^m(\mu)}{(1-\mu^2)} - 2\pi B_m \frac{dP_n^m}{d\mu}\right)\right] \\ & + 2\Omega[n(n-1)\epsilon_n^m \chi_{n-1}^m + (n+1)(n+2)\epsilon_{n+1}^m \chi_{n+1}^m - V_n^m] \end{aligned} \tag{99}$$

The divergence equation (22) and the continuity equation (23) can be transformed similarly to how we have treated the vorticity equation in the long, tedious process from (91) to (99). Here are the resulting equations, notationally translated from Washington & Parkinson without proof:

$$\begin{aligned} -n(n+1)\frac{\partial \chi_n^m}{\partial t} = & \frac{2\pi\xi_n^m}{a}\int_{-1}^{+1} d\mu \left(\frac{imB_m P_n^m}{1-\mu^2} + A_m \frac{dP_n^m}{d\mu}\right) \\ & - 2\Omega[n(n-1)\epsilon_n^m \Psi_{n-1}^m + (n+1)(n+2)\epsilon_{n+1}^m \Psi_{n+1}^m + U_n^m] \\ & + n(n+1)\left(\Phi_n^m + \int_{-1}^{+1} d\mu \frac{E_m P_n^m}{1-\mu^2}\right) \end{aligned} \tag{100}$$

$$\frac{\partial \Phi_n^m}{\partial t} = -\int_{-1}^{+1} d\mu \left(\frac{imC_m P_n^m}{1-\mu^2} - D_m \frac{dP_n^m}{d\mu}\right) + \bar{\Phi} n(n+1)\chi_n^m \tag{101}$$

Equations (99), (100), and (101) were the target of this whole writeup. Let us review what we have accomplished.

1. We began with the shallow water equations: three nonlinear partial differential equations, with three independent variables $u$, $v$, and $h$, in a domain of latitude, longitude, and time. Use of the shallow water equations is typical of heuristic formulations like this—it was complicated enough for these simple forms. Extension to the full primitive equations is discussed in Bourke et al. (1977), and the full resulting spectral equations can be found in Kiehl etal. (1996).
2. We end with equations whose only partial derivatives are in time, so we have prognostic equations that can be solved by temporal finite differencing. The prognostic variables are now a large set of spherical harmonic Fourier coefficients.
3. Along the way, we transformed the independent variable set from $u$, $v$, and $h$ into $\Psi$, $\chi$, and $\Phi$. What we lost in intuitive intelligibility was regained by the simplicity of calculating the Laplacian of a spherical harmonic.
4. Along the way, we deferred consideration of a lot of integrals. In order for this model to be remotely usable, there must be very effective and efficient ways of numerically integrating

$$\int_{-1}^{+1} d\mu\, f(\mu) \qquad \text{and} \qquad \int_{0}^{2\pi} d\lambda\, g(\lambda)$$

   Discussion of the Fast Fourier Transform and Gauss-Legendre Quadrature must now follow.

# Fast Fourier Transform and Gauss-Legendre Integration

Fitting the generalized Fourier coefficients for spherical harmonics to the various spatial fields requires that the model repeatedly evaluate the integrals needed, specifically

$$A_n^m = \int_0^{2\pi} d\lambda \int_{-1}^{+1} d\mu A(\mu, \lambda) Y_n^{m*}(\mu, \lambda) \tag{102}$$

Each of the integrals requires a special technique—Gauss-Legendre integration for the latitudinal ($d\mu$) integral, and the Fast Fourier Transform (FFT) for the longitudinal ($d\lambda$) integral. Inclusion of those in this writeup is a goal for future runs of the course.

# References

Bourke, W., 1972. An efficient, one-level primitive-equation spectral model. *Monthly Weather Review,*, **100:** 683–689. This paper is widely cited as the first spectral GCM experiment. Bourke later extended it to multiple vertical levels. His models became the Australian National GCM, from which all the NCAR CCM versions are descended.

Bourke, W, B. McAvaney, K. Puri, & R. Thurling, 1977. Global modeling of atmospheric flow by spectral methods. In J. Chang, ed., *Methods in Computational Physics 17: General Circulation Models of the Atmosphere.* New York, Academic Press. 268–324. Section II discusses the transform methods.

Haltiner, G. J., & R. T. Williams, 1980. *Numerical Prediction and Dynamic Meteorology.* New York, John Wiley. The most complete book on numerical modeling of weather. Chapter 5 covers finite-differences in great detail. Chapter 6 includes all spectral methods (finite-elements and spherical harmonics). Much of this writeup is based on section 6–6.

W. D. Collins and 12 others, 2004. Description of the NCAR Community Atmospheric Model (CAM3). NCAR Technical Note NCAR/TN464+STR. 214pp. NCAR publishes a technical note for each major version of each of its atmspheric model, and these include the complete spectral equations as actually used in the models. Not light reading.

Washington, W. M., & C. L. Parkinson, 2005. *An Introduction to Three-Dimensional Climate Modeling, 2nd Ed.* Mill Valley, Calif., University Science Books. The spectral model discussion is still good. The description of the spectral transform herein is simplified from Haltiner & Williams. They motivate the discussion by describing conventional Fourier sine series. Good appendix on Legendre polynomials and Gaussian integration.